Goldstein

1.9. The change induced on Lagrangian is,

$$L \to L + q\vec{\nabla}\psi \cdot \vec{v} + \frac{q}{c}\dot{\psi}\phi.$$

$$\Delta L = q\left[\vec{\nabla}\psi(\vec{r},t)\cdot\vec{v} + \frac{1}{c}\dot{\psi}\phi\right].$$

$$\Delta L = q\left[\frac{\partial}{\partial t}[\vec{\nabla}\psi\cdot\vec{r}] - \frac{\partial}{\partial t}(\vec{\nabla}\psi)\cdot\vec{r} + \frac{1}{c}\frac{\partial\psi}{\partial t}\phi\right]$$

$$= q\left[\frac{\partial}{\partial t}\right]\left[\vec{\nabla}\psi\cdot\vec{r} + \frac{1}{c}\phi\right] - \frac{\partial}{\partial t}(\vec{\nabla}\psi)\cdot\vec{r}.$$

The first term induces no change on the equation because it is the time derivative of arbitrary function over coordinates already obeying the old Lagrangian (problem 1.8). So it's sufficient to consider the second term only.

Redefine $\Delta L = \frac{\partial}{\partial t}(\vec{\nabla}\psi)\cdot\vec{r}$.

$$\frac{d}{dx}\Delta L = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}(\vec{\nabla}\psi)\right]\cdot\vec{r} + \frac{\partial}{\partial t}(\vec{\nabla}\psi)\cdot\vec{v}, \quad \frac{d}{d\dot{x}}\Delta L = 0.$$

Thus the eqm would be unaffected if $\psi$ obeys

$$\left[\vec{\nabla}\frac{\partial}{\partial t}(\vec{\nabla}\psi)\right]\cdot\vec{r} + \frac{\partial}{\partial t}(\vec{\nabla}\psi)\cdot\vec{v} = 0.$$